THE CENTERS OF GRAVITY OF THE ASSOCIAHEDRON AND OF THE PERMUTAHEDRON ARE THE SAME

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ABSTRACT. In this article, we show that Loday's realization of the associahedron has the the same center of gravity than the permutahedron. This proves an observation made by F. Chapoton.

We also prove that this result holds for the associahedron and the cyclohedron as realized by the first author and C. Lange.

1. INTRODUCTION.

In 1963, J. Stasheff discovered the associahedron [9, 10], a polytope of great importance in algebraic topology. The associahedron in $\mathbb{R}^n$ is a simple $n-1$-dimensional convex polytope. The classical realization of the associahedron given by S. Shnider and S. Sternberg in [7] was completed by J. L. Loday in 2004 [6]. Loday gave a combinatorial algorithm to compute the integer coordinates of the vertices of the associahedron, and showed that it can be obtained naturally from the classical permutahedron of dimension $n-1$. F. Chapoton observed that the centers of gravity of the associahedron and of the permutahedron are the same [6, Section 2.11]. As far as we know, this property of Loday's realization has never been proved.

In 2007, the first author and C. Lange gave a family of realizations of the associahedron that contains the classical realization of the associahedron. Each of these realizations is also obtained naturally from the classical permutahedron [4]. They conjectured that for any of these realizations, the center of gravity coincide with the center of gravity of the permutahedron. In this article, we prove this conjecture to be true.

The associahedron fits in a larger family of polytopes, *generalized associahedra*, introduced by S. Fomin and A. Zelevinsky in [3] within the framework of cluster algebras (see [2, 5] for their realizations).

In 1994, R. Bott and C. Taubes discovered the cyclohedron [1] in connection with knot theory. It was rediscovered independently by R. Simion [8]. In [4], the first author and C. Lange also gave a family of realizations for the cyclohedron, starting with the permutahedron of type B.

We also show that the centers of gravity of the cyclohedron and of the permutahedron of type B are the same.

The article is organized as follows. In §2, we first recall the realization of the permutahedron and how to compute its center of gravity. Then we compute the center of gravity of Loday's realization of the associahedron. In order to do this,

Date: October 2, 2009.

* This work is supported by FQRNT and NSERC. It is the result of a summer undergraduate research internship supported by LaCIM.

we partition its vertices into isometry classes of triangulations, which parameterize the vertices, and we show that the center of gravity for each of those classes is the center of gravity of the permutahedron.

In §3, we show that the computation of the center of gravity of any of the realizations given by the first author and C. Lange is reduced to the computation of the center of gravity of the classical realization of the associahedron. We do the same for the cyclohedron in §4.

We are grateful to Carsten Lange for allowing us to use some of the pictures he made in [4].

2. Center of gravity of the classical permutahedron and associahedron

2.1. The permutahedron. Let S_n be the symmetric group acting on the set $[n] = \{1, 2, \ldots, n\}$. The *permutahedron* $\mathsf{Perm}(S_n)$ is the classical $n-1$-dimensional simple convex polytope defined as the convex hull of the points

$$M(\sigma) = (\sigma(1), \sigma(2), \ldots, \sigma(n)) \in \mathbb{R}^n, \qquad \forall \sigma \in S_n.$$

The *center of gravity* (or *isobarycenter*) is the unique point G of $\mathbb{R}^n$ such that

$$\sum_{\sigma \in S_n} \overrightarrow{GM(\sigma)} = \overrightarrow{0}.$$

Since the permutation $w_0 : i \mapsto n+1-i$ preserves $\mathsf{Perm}(S_n)$, we see, by sending $M(\sigma)$ to

$$M(w_0\sigma) = (n+1-\sigma(1), n+1-\sigma(2), \ldots, n+1-\sigma(n)),$$

that the center of gravity is $G = (\frac{n+1}{2}, \frac{n+1}{2}, \ldots, \frac{n+1}{2})$.

2.2. Loday's realization. We present here the realization of the associahedron given by J. L. Loday [6]. However, instead of using planar binary trees, we use triangulations of a regular polygon to parameterize the vertices of the associahedron (see [4, Remark 1.2]).

2.2.1. *Triangulations of a regular polygon.* Let P be a regular $(n+2)$-gon in the Euclidean plane with vertices $A_0, A_1, \ldots, A_{n+1}$ in counterclockwise direction. A *triangulation of* P is a set of n noncrossing diagonals of P.

Let us be more explicit. A *triangle of* P is a triangle whose vertices are vertices of P. Therefore a side of a triangle of P is either an edge or a diagonal of P. A triangulation of P is then a collection of n distinct triangles of P with noncrossing sides. Any of the triangles in T can be described as $A_iA_jA_k$ with $0 \le i < j < k \le n+1$. Each $1 \le j \le n$ corresponds to a unique triangle $\Delta_j(T)$ in T because the sides of triangles in T are noncrossing.

Therefore we write $T = \{\Delta_1(T), \ldots, \Delta_n(T)\}$ for a triangulation T, where $\Delta_j(T)$ is the unique triangle in T with vertex A_j and the two other vertices A_i and A_k satisfying the inequation $0 \le i < j < k \le n+1$.

Denote by $\mathcal{T}_{n+2}$ the set of triangulations of P.

2.2.2. *Loday's realization of the associahedron.* Let T be a triangulation of P. The *weight* $\delta_j(T)$ of the triangle $\Delta_j(T) = A_iA_jA_k$, where $i < j < k$, is the positive number

$$\delta_j(T) = (j-i)(k-j).$$

The weight $\delta_j(T)$ of $\Delta_j(T)$ represents the product of the number of boundary edges of P between A_i and A_j passing through vertices indexed by smaller numbers than j with the number of boundary edges of P between A_j and A_k passing through vertices indexed by larger numbers than j.

The *classical associahedron* $\mathsf{Asso}(S_n)$ is obtained as the convex hull of the points

$$M(T) = (\delta_1(T), \delta_2(T), \dots, \delta_n(T)) \in \mathbb{R}^n, \quad \forall T \in \mathcal{T}_{n+2}.$$

We are now able to state our first result.

Theorem 2.1. *The center of gravity of* $\mathsf{Asso}(S_n)$ *is* $G = (\frac{n+1}{2}, \frac{n+1}{2}, \dots, \frac{n+1}{2})$.

In order to prove this theorem, we need to study closely a certain partition of the vertices of P.

2.3. **Isometry classes of triangulations.** As P is a regular $(n+2)$-gon, its isometry group is the dihedral group $\mathcal{D}_{n+2}$ of order $2(n+2)$. So $\mathcal{D}_{n+2}$ acts on the set $\mathcal{T}_{n+2}$ of all triangulations of P: for $f \in \mathcal{D}_{n+2}$ and $T \in \mathcal{T}_{n+2}$, we have $f \cdot T \in \mathcal{T}_{n+2}$. We denote by $\mathcal{O}(T)$ the orbit of $T \in \mathcal{T}_{n+2}$ under the action of $\mathcal{D}_{n+2}$.

We know that G is the center of gravity of $\mathsf{Asso}(S_n)$ if and only if

$$\sum_{T \in \mathcal{T}_{n+2}} \overrightarrow{GM(T)} = \overrightarrow{0}.$$

As the orbits of the action of $\mathcal{D}_{n+2}$ on $\mathcal{T}_{n+2}$ form a partition of the set $\mathcal{T}_{n+2}$, it is sufficient to compute

$$\sum_{T \in \mathcal{O}} \overrightarrow{GM(T)}$$

for any orbit $\mathcal{O}$. The following key observation implies directly Theorem 2.1.

Theorem 2.2. *Let* $\mathcal{O}$ *be an orbit of the action of* $\mathcal{D}_{n+2}$ *on* $\mathcal{T}_{n+2}$*, then* G *is the center of gravity of* $\{M(T) \,|\, T \in \mathcal{O}\}$. *In particular,* $\sum_{T\in\mathcal{O}} \overrightarrow{GM(T)} = \overrightarrow{0}$.

Before proving this theorem, we need to prove the following result.

Proposition 2.3. *Let* $T \in \mathcal{T}_{n+2}$ *and* $j \in [n]$*, then* $\displaystyle\sum_{f \in \mathcal{D}_{n+2}} \delta_j(f \cdot T) = (n+1)(n+2)$.

Proof. We prove this proposition by induction on $j \in [n]$. For any triangulation T', we denote by $a_j(T') < j < b_j(T')$ the indices of the vertices of $\Delta_j(T')$. Let H be the group of rotations in $\mathcal{D}_{n+2}$. It is well-known that for any reflection $s \in \mathcal{D}_{n+2}$, the classes H and sH form a partition of $\mathcal{D}_{n+2}$ and that $|H| = n+2$. We consider also the unique reflection $s_k \in \mathcal{D}_{n+2}$ which maps A_x to $A_{n+3+k-x}$, where the values of the indices are taken in modulo $n+2$. In particular, $s_k(A_0) = A_{n+3+k} = A_{k+1}$, $s_k(A_1) = A_k$, $s_k(A_{k+1}) = A_{n+2} = A_0$, and so on.

Basic step $j = 1$**:** We know that $a_1(T') = 0$ for any triangulation T', hence the weight of $\Delta_1(T')$ is $\delta_1(T') = (1-0)(b_1(T') - 1) = b_1(T') - 1$.

The reflection $s_0 \in \mathcal{D}_{n+2}$ maps A_x to A_{n+3-x} (where $A_{n+2} = A_0$ and $A_{n+3} = A_1$). In other words, $s_0(A_0) = A_1$ and $s_0(\Delta_1(T'))$ is a triangle in $s_0 \cdot T'$. Since

$$s_0(\Delta_1(T')) = s_0(A_0A_1A_{b_1(T')}) = A_0A_1A_{n+3-b_1(T')}$$

and $0 < 1 < n+3-b_1(T')$, $s_0(\Delta_1(T'))$ has to be $\Delta_1(s_0 \cdot T')$. In consequence, we obtain that

$$\delta_1(T') + \delta_1(s_0 \cdot T') = (b_1(T') - 1) + (n+3-b_1(T')-1) = n+1,$$

for any triangulation T'. Therefore

$$\sum_{f \in \mathcal{D}_{n+2}} \delta_1(f \cdot T) = \sum_{g \in H} \big((\delta_1(g \cdot T) + \delta_1(s_0 \cdot (g \cdot T))\big) = |H|(n+1) = (n+1)(n+2),$$

proving the initial case of the induction.

Inductive step: Assume that, for a given $1 \le j < n$, we have

$$\sum_{f \in \mathcal{D}_{n+2}} \delta_j(f \cdot T) = (n+1)(n+2).$$

We will show that

$$\sum_{f \in \mathcal{D}_{n+2}} \delta_{j+1}(f \cdot T) = \sum_{f \in \mathcal{D}_{n+2}} \delta_j(f \cdot T).$$

Let $r \in H \subseteq \mathcal{D}_{n+2}$ be the unique rotation mapping A_{j+1} to A_j. In particular, $r(A_0) = A_{n+1}$. Let T' be a triangulation of P. We have two cases:

Case 1. If $a_{j+1}(T') > 0$ then $a_{j+1}(T') - 1 < j < b_{j+1}(T') - 1$ are the indices of the vertices of the triangle $r(\Delta_{j+1}(T'))$ in $r \cdot T'$. Therefore, by unicity, $r(\Delta_{j+1}(T'))$ must be $\Delta_j(r \cdot T')$. Thus

$$\begin{aligned} \delta_{j+1}(T') &= (b_{j+1}(T') - (j+1))(j+1-a_{j+1}(T')) \\ &= \big((b_{j+1}(T') - 1) - j\big)(j - (a_{i+1}(T') - 1)) \\ &= \delta_j(r \cdot T'). \end{aligned}$$

In other words:

$$\begin{aligned} \sum_{\substack{f \in \mathcal{D}_{n+2}, \\ a_{j+1}(f \cdot T) \neq 0}} \delta_{j+1}(f \cdot T) &= \sum_{\substack{f \in \mathcal{D}_{n+2}, \\ a_{j+1}(f \cdot T) \neq 0}} \delta_j(r \cdot (f \cdot T)) \\ &= \sum_{\substack{g \in \mathcal{D}_{n+2}, \\ b_j(g \cdot T) \neq n+1}} \delta_j(g \cdot T). \end{aligned} \tag{1}$$

Case 2. If $a_{j+1}(T') = 0$, then $j < b_{j+1}(T') - 1 < n+1$ are the indices of the vertices of $r(\Delta_{j+1}(T'))$, which is therefore not $\Delta_j(r \cdot T')$: it is $\Delta_{b_{j+1}(T')-1}(r \cdot T')$. To handle this, we need to use the reflections s_j and s_{j-2}.

On one hand, observe that $j+1 < n+3+j-b_{j+1}(T')$ because $b_{j+1}(T') < n+1$. Therefore

$$s_j(\Delta_{j+1}(T')) = A_{j+1}A_0A_{n+3+j-b_{j+1}(T')} = \Delta_{j+1}(s_j \cdot T').$$

Hence

$$\begin{aligned} \delta_{j+1}(T') + \delta_{j+1}(s_j \cdot T') &= (j+1)(b_{j+1}(T') - (j+1)) \\ &\quad + (j+1)(n+3+j-b_{j+1}(T') - (j+1)) \\ &= (j+1)(n+1-j). \end{aligned}$$

On the other hand, consider the triangle $\Delta_j(r \cdot T')$ in $r \cdot T'$. Since

$$r(\Delta_{j+1}(T')) = A_jA_{b_{j+1}(T')-1}A_{n+1} = \Delta_{b_{j+1}(T')-1}(r \cdot T')$$

is in $r \cdot T'$, $[j, n+1]$ is a diagonal in $r \cdot T'$. Hence $b_j(r \cdot T') = n+1$. Thus $\Delta_j(r \cdot T') = A_{a_j(r \cdot T')} A_j A_{n+1}$ and $\delta_j(r \cdot T') = (j - a_j(r \cdot T'))(n+1-j)$. We have $s_{j-2}(A_j) = A_{n+1}$, $s_{j-2}(A_{n+2}) = A_j$ and $s_{j-2}(A_{a_j(r \cdot T')}) = A_{n+1+j-a_j(r \cdot T')} = A_{j-a_j(r \cdot T')-1}$ since $a_j(r \cdot T') < j$. Therefore $s_{j-2}(\Delta_j(r \cdot T')) = A_{j-a_j(r \cdot T')-1} A_j A_{n+1} = \Delta_j(s_{j-2} r \cdot T')$ and $\delta_j(s_{j-2} r \cdot T') = (a_j(r \cdot T') + 1)(n+1-j)$. Finally we obtain that

$$\begin{aligned} \delta_j(r \cdot T') + \delta_j(s_{j-2} r \cdot T') &= (j - a_j(r \cdot T'))(n+1-j) + (a_j(r \cdot T') + 1)(n+1-j) \\ &= (j+1)(n+1-j). \end{aligned}$$

Since $\{H, s_k H\}$ forms a partition of $\mathcal{D}_{n+2}$ for any k, we have

$$\begin{aligned} \sum_{\substack{f \in \mathcal{D}_{n+2}, \\ a_{j+1}(f \cdot T) = 0}} \delta_{j+1}(f \cdot T) &= \sum_{\substack{f \in H, \\ a_{j+1}(f \cdot T) = 0}} \big(\delta_{j+1}(f \cdot T) + \delta_{j+1}(s_j f \cdot T)\big) \\ &= \sum_{\substack{f \in H, \\ a_{j+1}(f \cdot T) = 0}} (j+1)(n+1-j) \\ &= \sum_{\substack{rf \in H, \\ b_j(rf \cdot T) = n+1}} \big(\delta_j(rf \cdot T) + \delta_j(s_{j-2} rf \cdot T)\big), \text{ since } r \in H \\ &= \sum_{\substack{g \in H, \\ b_j(g \cdot T) = n+1}} \delta_j(g \cdot T). \end{aligned} \tag{2}$$

We conclude the induction by adding Equations (1) and (2). □

Proof of Theorem 2.2. We have to prove that

$$\overrightarrow{u} = \sum_{T' \in \mathcal{O}(T)} \overrightarrow{GM(T')} = \overrightarrow{0}.$$

Denote by $\mathrm{Stab}(T') = \{f \in \mathcal{D}_{n+2} \mid f \cdot T' = T'\}$ the stabilizer of T', then

$$\sum_{f \in \mathcal{D}_{n+2}} M(f \cdot T) = \sum_{T' \in \mathcal{O}(T)} |\mathrm{Stab}(T')| M(T').$$

Since $T' \in \mathcal{O}(T)$, $|\mathrm{Stab}(T')| = |\mathrm{Stab}(T)| = \frac{2(n+2)}{|\mathcal{O}(T)|}$, we have

$$\sum_{f \in \mathcal{D}_{n+2}} M(f \cdot T) = \frac{2(n+2)}{|\mathcal{O}(T)|} \sum_{T' \in \mathcal{O}(T)} M(T').$$

Therefore by Proposition 2.3 we have for any $i \in [n]$

$$\sum_{T' \in \mathcal{O}(T)} \delta_i(T') = \frac{|\mathcal{O}(T)|}{2(n+2)}(n+1)(n+2) = \frac{|\mathcal{O}(T)|(n+1)}{2}. \tag{3}$$

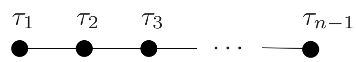
Denote by O the point of origin of $\mathbb{R}^n$. Then $\overrightarrow{OM} = M$ for any point M of $\mathbb{R}^n$. By Chasles' relation we have finally

$$\overrightarrow{u} = \sum_{T' \in \mathcal{O}(T)} \overrightarrow{GM(T')} = \sum_{T' \in \mathcal{O}(T)} (M(T') - G) = \sum_{T' \in \mathcal{O}(T)} M(T') - |\mathcal{O}(T)| G.$$

So the i^{th} coordinate of $\overrightarrow{u}$ is $\sum_{T' \in \mathcal{O}(T)} \delta_i(T') - \frac{|\mathcal{O}(T)|(n+1)}{2} = 0$, hence $\overrightarrow{u} = \overrightarrow{0}$ by (3). □

3. Center of gravity of generalized associahedra of type A and B

3.1. **Realizations of associahedra.** As a Coxeter group (of type A), S_n is generated by the simple transpositions $\tau_i = (i,\, i+1)$, $i \in [n-1]$. The Coxeter graph Γ_{n-1} is then



Let $\mathscr{A}$ be an orientation of Γ_{n-1}. We distinguish between *up* and *down* elements of $[n]$: an element $i \in [n]$ is *up* if the edge $\{\tau_{i-1}, \tau_i\}$ is directed from τ_i to τ_{i-1} and *down* otherwise (we set 1 and n to be down). Let $\mathsf{D}_{\mathscr{A}}$ be the set of down elements and let $\mathsf{U}_{\mathscr{A}}$ be the set of up elements (possibly empty).

The notion of up and down induces a labeling of the $(n+2)$-gon P as follows. Label A_0 by 0. Then the vertices of P are, in counterclockwise direction, labeled by the down elements in increasing order, then by $n+1$, and finally by the up elements in decreasing order. An example is given in Figure 1.

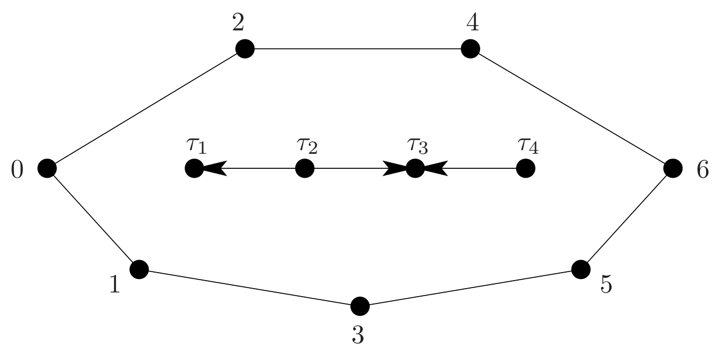


Figure 1. A labeling of a heptagon that corresponds to the orientation $\mathscr{A}$ of Γ_4 shown inside the heptagon. We have $\mathsf{D}_{\mathscr{A}} = \{1, 3, 5\}$ and $\mathsf{U}_{\mathscr{A}} = \{2, 4\}$.

We recall here a construction due to Hohlweg and Lange [4]. Consider P labeled according to a fixed orientation $\mathscr{A}$ of Γ_{n-1}. For each $l \in [n]$ and any triangulation T of P, there is a unique triangle $\Delta_l^{\mathscr{A}}(T)$ whose vertices are labeled by $k < l < m$. Now, count the number of edges of P between i and k, whose vertices are labeled by smaller numbers than l. Then multiply it by the number of edges of P between l and m, whose vertices are labeled by greater numbers than l. The result $\omega_l^{\mathscr{A}}(T)$ is called the *weight* of $\Delta_l^{\mathscr{A}}(T)$. The injective map

$$\begin{aligned} M_{\mathscr{A}} : \mathcal{T}_{n+2} &\longrightarrow \mathbb{R}^n \\ T &\longmapsto (x_1^{\mathscr{A}}(T), x_2^{\mathscr{A}}(T), \ldots, x_n^{\mathscr{A}}(T)) \end{aligned}$$

that assigns explicit coordinates to a triangulation is defined as follows:

$$x_j^{\mathscr{A}}(T) := \begin{cases} \omega_j^{\mathscr{A}}(T) & \text{if } j \in \mathsf{D}_{\mathscr{A}} \\ n+1-\omega_j^{\mathscr{A}}(T) & \text{if } j \in \mathsf{U}_{\mathscr{A}}. \end{cases}$$

Hohlweg and Lange showed that the convex hull $\mathsf{Asso}_{\mathscr{A}}(S_n)$ of $\{M_{\mathscr{A}}(T) \,|\, T \in \mathcal{T}_{n+2}\}$ is a realization of the associahedron with integer coordinates [4, Theorem 1.1]. Observe that if the orientation $\mathscr{A}$ is *canonic*, that is, if $\mathsf{U}_{\mathscr{A}} = \emptyset$, then $\mathsf{Asso}_{\mathscr{A}}(S_n) = \mathsf{Asso}(S_n)$.

The key is now to observe that the weight of $\Delta_j^{\mathscr{A}}(T)$ in T is precisely the weight of $\Delta_j(T')$ where T' is a triangulation in the orbit of T under the action of $\mathcal{D}_{n+2}$, as stated in the next proposition.

Proposition 3.1. *Let $\mathscr{A}$ be an orientation of Γ_{n-1}. Let $j \in [n]$ and let A_l be the vertex of P labeled by j. There is an isometry $r_j^{\mathscr{A}} \in \mathcal{D}_{n+2}$ such that:*

- *(i) $r_j^{\mathscr{A}}(A_l) = A_j$;*
- *(ii) the label of the vertex A_k is smaller than j if and only if the index i of the vertex $A_i = r_j^{\mathscr{A}}(A_k)$ is smaller than j.*

Moreover, for any triangulation T of P we have $\omega_j^{\mathscr{A}}(T) = \delta_j(r_j^{\mathscr{A}} \cdot T)$.

Proof. If $\mathscr{A}$ is the canonical orientation, then $r_j^{\mathscr{A}}$ is the identity, and the proposition is straightforward. In the following proof, we suppose therefore that $\mathsf{U}_{\mathscr{A}} \neq \emptyset$.

Case 1: Assume that $j \in \mathsf{D}_{\mathscr{A}}$. Let α be the greatest up element smaller than j and let $A_{\alpha+1}$ be the vertex of P labeled by α. Then by construction of the labeling, A_α is labeled by a larger number than j, and $[A_\alpha, A_{\alpha+1}]$ is the unique edge of P such that $A_{\alpha+1}$ is labeled by a smaller number than j. Denote by $\Lambda_{\mathscr{A}}$ the path from A_l to $A_{\alpha+1}$ passing through vertices of P labeled by smaller numbers than j. This is the path going from A_l to $A_{\alpha+1}$ in clockwise direction on the boundary of P.

By construction, $A_k \in \Lambda_{\mathscr{A}}$ if and only if the label of A_k is smaller than j. In other words, the path $\Lambda_{\mathscr{A}}$ consists of *all* vertices of P labeled by smaller numbers than j. Therefore the cardinality of $\Lambda_{\mathscr{A}}$ is $j+1$.

Consider $r_j^{\mathscr{A}}$ to be the rotation mapping A_l to A_j. Recall that a rotation is an isometry preserving the orientation of the plane. Then the path $\Lambda_{\mathscr{A}}$, which is obtained by walking on the boundary of P from A_l to $A_{\alpha+1}$ in clockwise direction, is sent to the path Λ obtained by walking on the boundary of P in clockwise direction from A_j and going through $j+1 = |\Lambda_{\mathscr{A}}|$ vertices of P. Therefore $\Lambda = \{A_0, A_1, \ldots, A_j\}$, thus proving the first claim of our proposition in this case.

Case 2: assume that $j \in \mathsf{U}_{\mathscr{A}}$. The proof is almost the same as in the case of a down element. Let α be the greatest down element smaller than j and let A_α be the vertex of P labeled by α. Then by construction of the labeling, $A_{\alpha+1}$ is labeled by a larger number than j, and $[A_\alpha, A_{\alpha+1}]$ is the unique edge of P such that A_α is labeled by a smaller number than j. Denote by $\Lambda_{\mathscr{A}}$ the path from A_l to A_α passing through vertices of P labeled by smaller numbers than j. This is the path going from A_α to A_l in clockwise direction on the boundary of P.

As above, $A_k \in \Lambda_{\mathscr{A}}$ if and only if the label of A_k is smaller than j. In other words, the path $\Lambda_{\mathscr{A}}$ consists of all the vertices of P labeled by smaller numbers than j. Therefore, again, the cardinality of $\Lambda_{\mathscr{A}}$ is $j+1$.

Let $r_j^{\mathscr{A}}$ be the reflection mapping A_α to A_0 and $A_{\alpha+1}$ to A_{n+1}. Recall that a reflection is an isometry reversing the orientation of the plane. Then the path $\Lambda_{\mathscr{A}}$,

which is obtained by walking on the boundary of P from A_α to A_l in clockwise direction, is sent to the path Λ obtained by walking on the boundary of P in clockwise direction from A_α and going through $j+1 = |\Lambda_{\mathscr{A}}|$ vertices of P. Therefore $\Lambda = \{A_0, A_1, \dots, A_j\}$. Hence $r_j^{\mathscr{A}}(A_l)$ is sent on the final vertex of the path Λ which is A_j, proving the first claim of our proposition.

Thus it remains to show that for a triangulation T of P we have $\omega_j^{\mathscr{A}}(T) = \delta_j(r_j^{\mathscr{A}} \cdot T)$. We know that $\Delta_j^{\mathscr{A}}(T) = A_k A_l A_m$ such that the label of A_k is smaller than j, which is smaller than the label of A_m. Write $A_a = r_j^{\mathscr{A}}(A_k)$ and $A_b = r_j^{\mathscr{A}}(A_m)$. Because of Proposition 3.1, $a < j < b$ and therefore

$$r_j^{\mathscr{A}}(\Delta_j^{\mathscr{A}}(T)) = A_a A_j A_b = \Delta_j(r_j^{\mathscr{A}} \cdot T).$$

So $(j-a)$ is the number of edges of P between A_l and A_k, whose vertices are labeled by smaller numbers than j. Similarly, $(b-j)$ is the number of edges between A_l and A_m, whose vertices are labeled by smaller numbers than j, and $(b-j)$ is the number of edges of P between A_l and A_m and whose vertices are labeled by larger numbers than j. So $\omega_l^{\mathscr{A}}(T) = (j-a)(b-j) = \delta_j(r_j^{\mathscr{A}} \cdot T)$. □

Corollary 3.2. *For any orientation $\mathscr{A}$ of the Coxeter graph of S_n and for any $j \in [n]$, we have*

$$\sum_{f \in \mathcal{D}_{n+2}} x_j^{\mathscr{A}}(f \cdot T) = (n+1)(n+2).$$

Proof. Let $r_j^{\mathscr{A}} \in \mathcal{D}_{n+2}$ be as in Proposition 3.1.

Suppose first that $j \in \mathsf{U}_{\mathscr{A}}$, then

$$\begin{aligned}
\sum_{f \in \mathcal{D}_{n+2}} x_i^{\mathscr{A}}(f \cdot T) &= 2(n+2)(n+1) - \sum_{f \in \mathcal{D}_{n+2}} \omega_i^{\mathscr{A}}(f \cdot T) \\
&= 2(n+2)(n+1) - \sum_{f \in \mathcal{D}_{n+2}} \delta_j(f r_j^{\mathscr{A}} \cdot T), \text{ by Proposition 3.1} \\
&= 2(n+2)(n+1) - \sum_{g \in \mathcal{D}_{n+2}} \delta_j(g^{\mathscr{A}} \cdot T), \text{ since } r_j^{\mathscr{A}} \in \mathcal{D}_{n+2} \\
&= (n+1)(n+2), \text{ by Proposition 2.3}
\end{aligned}$$

If $i \in \mathsf{D}_{\mathscr{A}}$, the result follows from a similar calculation. □

3.2. **Center of gravity of associahedra.**

Theorem 3.3. *The center of gravity of $\mathsf{Asso}_{\mathscr{A}}(S_n)$ is $G = (\frac{n+1}{2}, \frac{n+1}{2}, \dots, \frac{n+1}{2})$ for any orientation $\mathscr{A}$.*

By following precisely the same arguments as in §2.3, we just have to show the following generalization of Theorem 2.2.

Theorem 3.4. *Let $\mathcal{O}$ be an orbit of the action of $\mathcal{D}_{n+2}$ on $\mathcal{T}_{n+2}$, then G is the center of gravity of $\{M_{\mathscr{A}}(T) \mid T \in \mathcal{O}\}$. In particular, $\sum_{T \in \mathcal{O}} \overrightarrow{GM_{\mathscr{A}}(T)} = \overrightarrow{0}$.*

Proof. The proof is entirely similar to the proof of Theorem 2.2, using Corollary 3.2 instead of Proposition 2.3. □

4. Center of gravity of the cyclohedron

4.1. **The type B-permutahedron.** The hyperoctahedral group W_n is defined by $W_n = \{\sigma \in S_{2n} \,|\, \sigma(i)+\sigma(2n+1-i) = 2n+1,\ \forall i \in [n]\}$. The *type B-permutahedron* $\mathsf{Perm}(W_n)$ is the simple n-dimensional convex polytope defined as the convex hull of the points

$$M(\sigma) = (\sigma(1), \sigma(2), \dots, \sigma(n)) \in \mathbb{R}^{2n}, \qquad \forall \sigma \in W_n.$$

As $w_0 = (2n, 2n-1, \dots, 3, 2, 1) \in W_n$, we deduce from the same argument as in the case of $\mathsf{Perm}(S_n)$ that the center of gravity of $\mathsf{Perm}(W_n)$ is

$$G = (\frac{2n+1}{2}, \frac{2n+1}{2}, \dots, \frac{2n+1}{2}).$$

4.2. **Realizations of the associahedron.** An orientation $\mathscr{A}$ of Γ_{2n-1} is *symmetric* if the edges $\{\tau_i, \tau_{i+1}\}$ and $\{\tau_{2n-i-1}, \tau_{2n-i}\}$ are oriented in *opposite directions* for all $i \in [2n-2]$. There is a bijection between symmetric orientations of Γ_{2n-1} and orientations of the Coxeter graph of W_n (see [4, §1.2]). A triangulation $T \in \mathcal{T}_{2n+2}$ is *centrally symmetric* if T, viewed as a triangulation of P, is centrally symmetric. Let $\mathcal{T}^B_{2n+2}$ be the set of the centrally symmetric triangulations of P. In [4, Theorem 1.5] the authors show that for any symmetric orientation $\mathscr{A}$ of Γ_{2n-1}. The convex hull $\mathsf{Asso}_{\mathscr{A}}(W_n)$ of $\{M_{\mathscr{A}}(T) \,|\, T \in \mathcal{T}^B_{2n+2}\}$ is a realization of the cyclohedron with integer coordinates.

Since the full orbit of symmetric triangulations under the action of $\mathcal{D}_{2n+2}$ on triangulations provides vertices of $\mathsf{Asso}_{\mathscr{A}}(W_n)$, and vice-versa, Theorem 3.4 implies the following corollary.

Corollary 4.1. *Let $\mathscr{A}$ be a symmetric orientation of Γ_{2n-1}, then the center of gravity of $\mathsf{Asso}_{\mathscr{A}}(W_n)$ is $G = (\frac{2n+1}{2}, \frac{2n+1}{2}, \dots, \frac{2n+1}{2})$.*

References

[1] R. Bott and C. Taubes, *On the self-linking of knots*, J. Math. Phys. **35** (1994), 5247–5287.
[2] F. Chapoton, S. Fomin and A. Zelevinsky, *Polytopal realizations of generalized associahedra*, Canad. Math. Bull. **45** (2003), 537–566.
[3] S. Fomin and A. Zelevinsky, *Y-systems and generalized associahedra*, Annals of Mathematics **158** (2003), 977–1018.
[4] C. Hohlweg and C. Lange, *Realizations of the Associahedron and Cyclohedron*, Discrete Comput Geom **37** (2007), 517–543.
[5] C. Hohlweg, C. Lange, H. Thomas, *Permutahedra and Generalized Associahedra*, `arXiv:math.CO/0709.4241`.
[6] J.-L. Loday, *Realization of the Stasheff polytope*, Arch. Math. **83** (2004), 267–278.
[7] S. Shnider and S. Sternberg, *Quantum groups: From coalgebas to Drinfeld algebras*, Graduate texts in mathematical physics, International Press, 1994.
[8] R. Simion, *A type-B associahedron*, Adv. Appl. Math. **30** (2003), 2–25.
[9] J. Stasheff, *Homotopy associativity of H-spaces I, II*, Trans. Amer. Math. Soc. **108** (1963), 275–312.
[10] J. Stasheff, *From operads to "physically" inspired theories*, Operads: Proceedings of Renaissance Conferences (Hartford, CT/Luminy, 1995), 53–81, Contemp. Math., 202, Americ. Math. Soc. (1997).

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